$$W(t) = W(t_0) \sum_{n=0}^{\infty} l_n \Delta t^n \qquad (\Delta t = t - t_0) \tag{3.8}$$

This series converges rapidly for permissible trajectories by virtue of the boundedness of $W(t)$.

Practically, with many real trajectories it is sufficient to retain three terms only,

$$W = W(t_0)(l_0 + l_1\Delta t + l_2\Delta t^2), \qquad W(t_0) = (j_0 + j_1 t_0 + \ldots + j_8 t_0^8)^{1/2} \tag{3.9}$$

Computing the coefficients l_0, l_1, l_2 of this polynomial, we obtain

$$l_0 = 1, \; l_1 = \frac{1}{2W^2(t_0)} \sum_{n=0}^{7} (n+1)\, j_{n+1} t_0^n$$

$$l_2 = \frac{1}{4W^2(t_0)} \sum_{n=0}^{6} (n+1)(n+2)\, j_{n+2} t_0^n - \frac{1}{2} l_1^2$$

Integrating Eq. (1.4) with allowance for (3.9), we obtain the apparent velocity expended on control,

$$v(T) = TW(t_0)[l_0 + l_1(^1/_2 T - t_0) + l_2(^1/_3 T^2 - t_0 T + t^2_0)] \tag{3.10}$$

The permissibility of a chosen trajectory of craft motion with fixed ends can be verified by substituting (3.6), (3.10) into inequalities (1.5).

BIBLIOGRAPHY

1. Mamatkazin, D. A., Some laws of spacecraft orbit control. PMM Vol. 32, №3, 1968.

Translated by A. Y.

ON THE AUTOOSCILLATIONS OF GYROSCOPIC STABILIZERS

PMM Vol. 34, №2, 1970, pp. 380-384

Iu. N. BIBIKOV and A. M. LESTEV
(Leningrad)

(Received June 10, 1969)

The system of third-order differential equations describing the motion of a single-axis gyro stabilizer with a floating integrating gyro is investigated. The stabilizer motor is controlled by means of a contact device with a dead zone δ. It is shown that for a sufficiently small δ the system has a closed trajectory corresponding to the autooscillations of the gyro stabilizer. The domain of immersion of the closed trajectory in the phase space is specified.

The autooscillations of gyro stabilizers were investigated in [1-4]. The author of [1, 2] analyzed the motion of a gyro stabilizer in the case of a relay-type stabilizer motor control. He determined the parameters and investigated the stability of the periodic motion by the method of point transformations. The author of [3, 4] treated the problem by the harmonic linearization method of E. P. Popov in conjunction with electronic modelling. The primary emphasis in these studies was on computing the periodic motion. In the theory of gyroscopic instruments employing autooscillatory operating modes it is especially important to investigate the conditions of existence of closed trajectories of the differential equations of gyro system motion, to prove the existence of these trajectories, and to localize them. The present paper deals with these matters in the case of a single-axis gyro stabilizer with a float-type integrator.

1. Let us suppose that the gyro stabilizer is mounted on a fixed base. We neglect the elastic pliability of the structural elements of the gyro stabilizer and the dry friction forces in the suspension supports and assume that the stabilizer is driven by a two-phase induction motor. The motor is controlled by means of a contact device with a dead zone δ. Under these assumptions the differential equations of the gyro stabilizer can be written as

$$A\ddot{\psi} + H\dot{\vartheta} = -M(\vartheta) - n_2\dot{\psi}, \qquad B\ddot{\vartheta} - H\dot{\psi} = -n_1\dot{\vartheta} \tag{1.1}$$
$$M(\vartheta) = M_0 \operatorname{sign} \vartheta \ (|\vartheta| \geqslant \delta), \qquad M(\vartheta) = 0 \ (|\vartheta| < \delta)$$

Here φ is the angle of rotation of the outer gimbal, ϑ is the angle of rotation of the housing, A is the sum of moments of the moving parts of the gyro stabilizer which are applied to the stabilizer axis, B is the sum of moments of inertia of the housing and gyro wheel relative to the housing axis, H is the kinetic moment of the gyroscope, n_1, n_2 are the damping factors, and $M(\vartheta)$ is the moment generated by the stabilizer motor.

Let us introduce the notation

$$\lambda_1 = \frac{H}{B}, \quad \lambda_2 = \frac{H}{A}, \quad \nu_1 = \frac{n_1}{B}, \quad \nu_2 = \frac{n_2}{A}, \quad f(\vartheta) = \frac{M(\vartheta)}{A}, \quad f_0 = \frac{M_0}{A}$$

and the new variables

$$\psi^{\cdot} = \xi, \ \vartheta^{\cdot} = \eta, \ \vartheta = \zeta$$

Equations (1.1) can be rewritten as

$$\xi^{\cdot} = -\nu_2\xi - \lambda_2\eta - f(\zeta), \qquad \eta^{\cdot} = \lambda_1\xi - \nu_1\eta, \quad \zeta^{\cdot} = \eta \tag{1.2}$$
$$f(\zeta) = f_0 \operatorname{sign} \zeta \ (|\zeta| \geqslant \delta), \quad f(\zeta) = 0 \ (|\zeta| < \delta)$$

The parameters of gyro stabilizers with float-type gyroscopes are usually such that the condition

$$(\nu_1 + \nu_2)^2 \geqslant 4(\lambda_1\lambda_2 + \nu_1\nu_2) \tag{1.3}$$

holds.

We propose to prove the existence of a closed trajectory of system (1.2) and to determine the domain containing the closed trajectory.

2. The transformation

$$x = \lambda_1\xi + \nu_2\eta + (\lambda_1\lambda_2 + \nu_1\nu_2)\zeta, \quad y = \eta, \quad z = \zeta \tag{2.1}$$

converts system (1.2) into

$$x^{\cdot} = -\varphi(z), \quad y^{\cdot} = x - \alpha y - \beta z, \quad z^{\cdot} = y \tag{2.2}$$

where

$$\alpha = \nu_1 + \nu_2, \quad \beta = \lambda_1\lambda_2 + \nu_1\nu_2, \quad \varphi(z) = \lambda_1 f(z)$$

Let us take (2.2) as our initial system, stipulating that

$$\alpha > 0, \quad \beta > 0 \qquad \varphi(z) = \varphi_0 \operatorname{sign} z \ (|z| \geqslant \delta), \ \varphi_0 = \lambda_1 f_0, \ \ \varphi(z) = 0 \ (|z| < \delta)$$

From now on we assume the validity of the inequality

$$\alpha^2 \geqslant 4\beta \tag{2.3}$$

which is equivalent to (1.3). Next, we introduce the constant

$$\mu = {}^1/_2 (\alpha + \sqrt{\alpha^2 - 4\beta}), \qquad {}^1/_2\, \alpha \leqslant \mu < \alpha$$

Let us consider the conical surface K_1 (Fig. 1) formed by the following planes:

$$y^{\cdot}=0,\quad 0\leqslant \beta z\leqslant x,\quad z=0,\quad x\geqslant \mu y,\quad y\leqslant 0$$
$$x-\mu y-\beta z=0,\quad y\leqslant 0$$

We denote the surface symmetric to K_1 with respect to the origin by K_2. Let S be the complement of the domains bounded by K_1 and K_2 to the entire space, and let R be the intersection of S with the plane $z=0$ for $x\geqslant 0$. The domain R is defined by the inequalities $0\leqslant y\leqslant \mu^{-1}x,\ x\geqslant 0$.

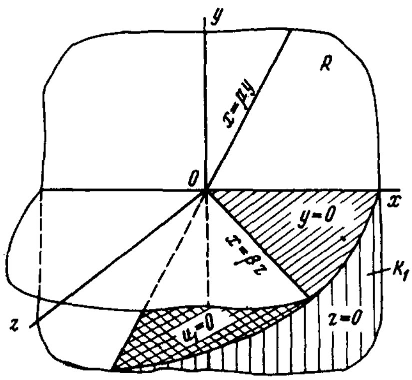

Fig. 1

Let $L^+(p,\,t)$ denote a positive semitrajectory of system (2.2) which passes through the point p for $t=0$.

Lemma 2.1. If $p\in S$, then $L^+(p,\,t)\in S$.

To prove this we need merely show that the trajectories of system (2.2) cannot intersect the surfaces K_1 and K_2 within the domains bounded by these surfaces. In the case of K_1 this can be inferred from the following relations, where $[u^{\cdot}]_{(2.2)}$ denotes the derivative of system (2.2) with respect to t:

$$[y^{\cdot}]_{(2.2)}=x-\beta z\quad (0\leqslant \beta z\leqslant x,\ \ y=0)$$
$$[z^{\cdot}]_{(2.2)}=y\leqslant 0\qquad (y\leqslant 0)$$

Let $u_1=x-\mu y-\beta z$. Then

$$[u_1^{\cdot}]_{(2.2)}=-\varphi(z)-\mu x+y\,(\mu\alpha-\beta)+\mu\beta z$$

For $u_1=0,\ z\geqslant 0$ we have

$$[u_1^{\cdot}]_{(2.2)}=-\varphi(z)-y(\mu^2-\mu\alpha+\beta)=-\varphi(z)\leqslant 0$$

The argument for the surface K_2 is similar. The lemma has been proved.

Lemma 2.2. There exists a bounded domain $Q\subset S$ such that if $p\in Q$, then $L^+(p,\,t)\subset Q$.

The proof consists in the construction of the surface Γ bounding the domain Q. Let us consider the plane

$$u_2=x-{}^1/_4\,\alpha y-\beta z=-c,\qquad c>0 \tag{2.4}$$

The plane (2.4) intersects the plane $u_1=0$ along the straight line along which $y=-c(\mu-{}^1/_4\alpha)^{-1}$ and the plane $z=0$ along the straight line $x-{}^1/_4\,\alpha y=-c$.

Let us show that

$$[u_2^{\cdot}]_{(2.2)}>0\ \text{ in the domain } u_2=-c,\ |y|\leqslant c(\mu-{}^1/_4\alpha)^{-1} \tag{2.5}$$

if c is sufficiently large. For $u_2=-c$ we have

$$[u_2^{\cdot}]_{(2.2)}=-\varphi(z)+{}^1/_4\alpha c+({}^3/_{16}\,\alpha^2-\beta)y$$

If $\beta\leqslant{}^3/_{16}\,\alpha^2$, then (2.5) is valid for

$$c>c_1=\varphi_0(4\mu-\alpha)\,(4\beta+\mu\alpha-\alpha^2)^{-1}$$

For $\beta>{}^3/_{16}\alpha^2$ inequality (2.5) is fulfilled if

$$c>c_2=\varphi_0(4\mu-\alpha)\,({}^1/_2\alpha^2+\mu\alpha-4\beta)^{-1}$$

Taking $c>c_1$ or $c>c_2$ according to the sign of $\beta-{}^3/_{16}\alpha^2$, we find that the part of the plane $u_2=-c$ for which $|y|<c(\mu-{}^1/_4\alpha)^{-1},\ z\geqslant 0$ is intersected by the trajectories of system (2.2) in the direction of increasing x.

We note, moreover, that since $\mu < \alpha$, it follows that $[y^\cdot]_{(2.2)} < 0$ in the domain ~ $u_1 <0> 0$.

In similar fashion we can show that

$$[u_2^\cdot]_{(2.2)} < 0, \quad \text{if} \quad u_2 = c, \ |y| \leqslant c(\mu - {}^1/_4\alpha)^{-1}$$
$$[y^\cdot]_{(2.2)} > 0 \quad \text{for} \quad u_1 \geqslant 0, \ y < 0$$

We form the surface Γ out of the planes

$$u_2 = \pm c, \qquad y = \mp c(\mu - {}^1/_4\alpha)^{-1}$$

in the indicated domains, adding to them the planes

$$x = -\mu c(\mu - {}^1/_4\alpha)^{-1}, \quad z \geqslant 0 \ \text{ and } \ x = \mu c(\mu - {}^1/_4\alpha)^{-1}, \ z \leqslant 0$$

In order to make Γ closed we must add the triangle T_1 in the plane $z = 0$ bounded by the negative semiaxis and the straight lines $x = -\mu c(\mu - {}^1/_4\alpha)^{-1}$, $x - {}^1/_4\,\alpha y = -c$, the triangle T_2 symmetric to T_1, and the corresponding elements of the surfaces K_1 and K_2.

The above analysis implies that the surface Γ is intersected by the trajectories of system (2.2) inside the domain Q bounded by Γ, so that Q has the property required by Lemma 2.2. The lemma has been proved.

Theorem. If condition (2.3) is fulfilled, the system (2.2) has a closed trajectory immersed in Q provided that δ is sufficiently small.

Proof. Let us denote by R_1 that part of the sector R which belongs to Q and for which $x \geqslant \beta z_0$, where $z_0 > \delta$ is a certain constant defined below.

Let us consider the straight line l in the plane xz. This line is defined by equation

$$x - \frac{2\beta z_0}{h} z = -1/h\ \beta z_0 (z_0 + \delta), \qquad y = 0 \qquad (h = z_0 - \delta > 0)$$

Next, let us construct the plane P with the equation

$$u_3 = x - \frac{2\alpha z_0}{h}\ y - \frac{2\beta z_0}{h}(z - z_0 - \delta) = 0$$

which passes through the straight line l and the point $A(\beta z_0, \beta h/\alpha, \delta)$ lying in the plane $y^\cdot = x - \alpha y - \beta z = 0$.

In the plane P we have

$$[u_3^\cdot]_{(2.2)} = -\varphi(z) - \frac{2\beta z_0}{h}\ y + \frac{\alpha(z_0 + \delta)}{h}(\beta z_0 - x) \tag{2.6}$$

Setting $h = k\delta$, we have $z_0 = (k + 1)\delta$, and (2.6) implies that

$$[u_3^\cdot]_{(2.2)} < 0 \qquad (z \geqslant \delta, \ |x| \leqslant \beta z_0)$$

in the plane P, provided that the inequality

$$\delta < \frac{k\varphi_0}{2\alpha\beta}(k+1)^{-1}(k+2)^{-1} \tag{2.7}$$

is valid.

Geometrically this means that any trajectory which passes through a point lying under the plane P (Fig. 2) for $|x| \leqslant \beta z_0$, $z \geqslant \delta$ intersects the plane $y = 0$ in the domain

$$x < \frac{2\beta z_0}{h} z - \frac{\beta z_0}{h}(z_0 + \delta), \qquad z \geqslant \delta, \qquad x \leqslant \beta z_0$$

with increasing time.

We shall show that a trajectory passing through the point B $(\beta z_0, 0, 0)$ has this property. System (2.2) is integrable in the domain $0 \leqslant z \leqslant \delta$ because

$$x(t) = \beta z_0, \qquad \frac{dy}{dz} = -\frac{1}{y}\,[\alpha y + \beta(z - z_0)] \tag{2.8}$$

in this domain.

Taking account of the initial values, we obtain

$$y^2 + \alpha y u + \beta u^2 = \beta z_0^2 \left[\frac{y(\alpha + \sqrt{\Delta}) + 2\beta u}{y(\alpha - \sqrt{\Delta}) + 2\beta u}\right]^{\varkappa}$$

$$u = z - z_0, \quad \varkappa = \alpha / \sqrt{\Delta}, \qquad \Delta = \alpha^2 - 4\beta$$

Let $y = y_0$ for $z = \delta$. This enables us to determine y_0 from the equation

$$y_0^2 + \alpha h y_0 + \beta h^2 = \beta z_0^2 \left[\frac{y_0(\alpha + \sqrt{\Delta}) - 2h\beta}{y_0(\alpha - \sqrt{\Delta}) - 2h\beta}\right]^{\varkappa}$$

Let us require that $y_0 \geqslant \beta h / \alpha$. This inequality is valid if

$$k\left(\frac{\sqrt{\beta}}{\alpha}\exp\left\{\frac{\varkappa}{2}\ln\frac{\alpha + \sqrt{\Delta}}{\alpha - \sqrt{\Delta}}\right\} - 1\right) \leqslant 1 \tag{2.9}$$

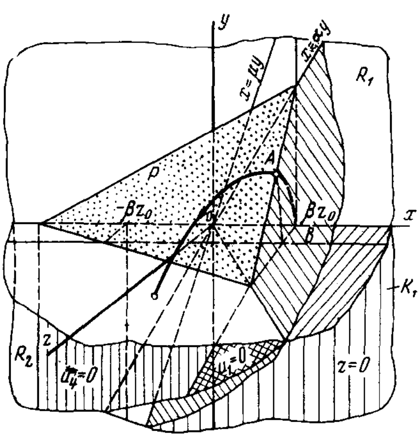

Fig. 2

From now on we assume that k satisfies inequality (2.9). The trajectory in question then lies above the plane P for $z = \delta$. The theorem has been proved.

Now let us consider an arbitrary trajectory passing through R_1. By virtue of (2.8) the derivative dy / dz increases together with z_0, so that $y_0 \geqslant \beta h / \alpha$ as before.

Moreover,

$$[y^{\cdot}]_{(2.2)} = -\varphi(z) - \beta y < 0$$

$$(y^{\cdot} = 0, \; y > 0, \; z \geqslant \delta)$$

The trajectories in these domains can intersect the plane $y^{\cdot} = 0$ only in the direction of decreasing $y^{\cdot}$, or, equivalently, of decreasing x. Their points of intersection with the plane $y = 0$ therefore lie

in the domain

$$x \leqslant \frac{2\beta z_0}{h} - \frac{\beta z_0(z_0 + \delta)}{h}, \qquad z \geqslant \delta \qquad (x \leqslant \beta z_0)$$

in the domain

$$x \leqslant \beta z \quad (x > \beta z_0)$$

Now let us consider the plane

$$u_4 = x - \frac{2\beta z_0}{h}(z - z_0 - \delta) = 0$$

in the domain Q for $y \leqslant 0$, $z \geqslant \delta$. This plane passes through the straight line l. If $u_4 = 0$, then

$$[u_4^{\cdot}]_{(2.2)} = -\varphi(z) - \frac{2\beta z_0}{h} y < 0 \tag{2.10}$$

in the domain Q for $z \geqslant 0$, $y \leqslant 0$, provided that

$$|y| < \frac{h\varphi_0}{2\beta z_0} = \frac{k\varphi_0}{2\beta(k+1)}$$

But in the domain Q in the plane $u_4 = 0$ we have

$$|y| < \frac{\beta}{\mu}(z_0 + \delta) = \frac{\beta\delta}{\mu}(k+2) \qquad (y \leqslant 0,\ z \geqslant \delta)$$

Hence, inequality (2.10) is valid if

$$\delta < \frac{k\mu\varphi_0}{2\beta^2}(k+1)^{-1}(k+2)^{-1} \tag{2.11}$$

The derivative $x^{\cdot} = 0$ for $0 \leqslant z \leqslant \delta$, and the straight line l has the abscissa $-\beta z_0$ for $z = \delta$. Hence, the trajectories of system (2.2) which originate in the domain R_1 intersect the domain R_2 of the plane xy (which is symmetric to R_1) with increasing time. Later on the trajectories of system (2.2) again intersect R_1, thereby defining the homeomorphism T with the property $T(R_1) \subset R_1$ on R_1. Hence, T has a fixed point on R_1 through which the closed trajectory of system (2.2) immersed in the domain Q must pass. The theorem has been proved.

Note Expressions (2.7), (2.9) and (2.11) imply that it is advisable to set $k = \sqrt{2}$. Let us consider a numerical example. Let $A = 1000$ g cm s^2, $B = 3.5$ g cm s^2, $H = 1000$ g cm s, $n_1 = 400$ g cm s, $n_2 = 2800$ g cm s, $M_0 = 5000$ g cm, and $k = \sqrt{2}$. A periodic motion exists for $\delta < 5.9$ angular minutes.

In conclusion we note that the above theorem remains valid in the case where the moment $M(\vartheta)$ of the stabilizer motor has a piecewise-linear characteristic with saturation, i.e. in the case

$$\varphi(z) = \varphi_0 \quad (|z| > \delta), \qquad \varphi(z) = \varphi_0\delta^{-1}z \quad (|z| \leqslant \delta)$$

In fact, the proofs of Lemmas 2.1 and 2.2 remain unchanged. Moreover, let us set

$$\delta < \varphi_0/\alpha\beta \tag{2.12}$$

Fulfillment of inequality (2.12) implies that the characteristic equation of the linear system (for $|z| \leqslant \delta$) has a single negative root ρ_1 and two complex roots with a positive real part. The direction of arrival of the trajectories at the origin as $t \to \infty$ is defined by the vector $(\varphi_0/\delta, -\rho_1^2, -\rho_1)$, i.e. it lies inside the cones K_1 and K_2. It remains for us to note that for $|z| \leqslant \delta$ the right circular cylinders whose axis coincides with the indicated direction intersect the trajectories as they expand.

BIBLIOGRAPHY

1. Roitenberg, Ia. N., Self-excited oscillations of gyroscopic stabilizers. PMM Vol. 11, №2, 1947.
2. Roitenberg, Ia. N., Gyroscopes. Moscow, "Nauka", 1966.
3. Chernikov, S. A., Symmetric autooscillations of a gyro stabilizer. Izv. Akad. Nauk SSSR, Energetika i Avtomatika №6, 1960.
4. Chernikov, S. A., Nonsymmetric autooscillations of a gyro stabilizer. Izv. Akad. Nauk SSSR, OTN, Energetika i Avtomatika №2, 1962.
5. Rivkin, S. S., Theory of Gyroscopic Devices, Part 1 and 2. Leningrad, Sudostroenie, 1962, 1964.

Translated by A.Y.